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On Free Products of Torsion Free Abelian Groups

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Let G be a group. For $x,y \in G$ the commutator of x and y is

$$(x,y) = x^{-1} y^{-1} x y.$$

The lower central series of G is a sequence of normal subgroups of G defined inductively as follows: $G_1 = G$, and having defined G_n (for n a positive integer), G_{n+1} is defined as the group generated by all commutators of (x,y) where $x \in G_n$ and $y \in G$. In this paper the lower central series is studied for a group J which is a free product of a finite number of torsion free Abelian groups. In particular, the quotient groups J_n / J_{n+1} are completely determined.

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CONTENTS

ON FREE PRODUCTS OF TORSION FREE ABELIAN GROUPS

1. INTRODUCTION

In a previous paper [1], the quotient groups of the lower central series $\bar{G}_n = G_n / G_{n+1}$ were studied. There the group G was assumed to be a free product of a finite number of finitely generated Abelian groups and G_n denoted the n^{th} subgroup of the lower central series of G. Here we give an improved proof of a complicated lemma which first appeared in [1] (in particular, Lemma 4.4[1]). The proof given here, especially for property (iii) of the conclusion of that lemma, is a significant simplification of that which appears in [1]. We observe that one of the consequences of Lemma 4.4[1] is to give a set of free generators for the lower central quotients in the case where the free factors are torsion free (i.e., G = J in the terminology of [1]. Moreover the free generators are the J-basic commutators also using the terminology of Definition 4.1 of [1].). The author only thought of this simplification after the publication of [1]. Furthermore, the improved proof uses results from [2] and [3].

In this paper, notation, terminology, results, references, and equations of [1] will be employed. Furthermore, the numbering, a.b,of any definition, equation, etc., of [1] will correspond here to a.b-I. For example,Lemma 2.1-I will mean Lemma 2.1 of [1].

2. PRELIMINARIES

In order to carry out our goal, we need to give some preliminary machinery.

Lemma 2.1. (cf. Corollary 5.1 [3]) Let S be the subgroup of

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F generated by a finite set Σ of F-simple commutators (see Definition 3.6-I) of weight > 1. Let $U = \{u_1, u_2, \ldots, u_m\} \subseteq \Sigma$. Let $V = \{v_1, v_2, \ldots, v_m\}$ be a set of m nontrivial elements v_i of F such that $W(v_i) \geq W(u_i)$ for $1 \leq i \leq m$. (Here we are using the notation of Definition 2.1-I for weight.) Let Ψ be the mapping of Σ given by

$$u_i \longrightarrow v_i \text{ if } u_i \in U$$

$$c_i \longrightarrow c_i \text{ if } c_i \notin U.$$

Then Ψ extends to a homomorphism, also denoted by Ψ, of S into F such that if $u \in S$ and $W(u) = \omega > 0$, then its image $v = \Psi(u) \in F_\omega$.

We also will require

Lemma 2.2. (cf. Theorem 1.1[2]) Let c be the F-simple basic commutator of weight n > 1,

$$c = (c_{j_1}, c_{j_2}, \ldots, c_{j_n}). \qquad (2.1)$$

Let d be a generator of F such that $d < c_{j_n}$. Let $v = (c_{j_2}, d, e_1, e_2, \ldots, e_{n-1})$ be such that $e_1 = c_{j_1}$ for $n = 2$, but $e_1 \leq e_2 \leq \ldots \leq e_{n-1}$ is a rearrangement of $c_{j_1}, c_{j_3}, \ldots, c_{j_n}$, for $n > 2$. (Note that v is F-simple of weight n + 1 if $d < c_{j_2}$.) Then the following identity holds in F

$$(c,d) = \begin{cases} \Pi_1 \, M\,(c,d)\, \Pi_2 & \text{if } d \geq c_{j_2} \text{ (Case I)} \\ \Pi_1 \, v^{-1} \, \Pi_2 \, M\,(c,d)\, \Pi_3 & \text{if } d < c_{j_2} \text{ (Case II)} \end{cases} \qquad (2.2)$$

where the $\Pi_j (j = 1,2,3)$ are either $= 1$ or are words in finitely many F-simple commutators u_i such that

(i) $1 < W(u_i) < n + 1$;

(ii) if $u_i = (v_1, v_2, \ldots, v_t)$, then $v_1, v_2, \ldots, v_t$ is a rearrangement of a subsequence of $c_{j_1}, \ldots, c_{j_n}, d$.

3. STATEMENT AND PROOF OF THE MAIN RESULT

We are now ready to state and prove our main result

Theorem 3.1. (See Lemma 4.4-I.) Let c be F-simple as in (2.1). Suppose that c satisfies Criterion 1 but not Criterion 3. (See Definition 4.1-I.) Let t be the largest integer such that c_{j_t} occurs in c and $(c_{j_t}, c_{j_1}) = 1$ in J. (The hypothesis that c does not satisfy Criterion 3 guarantees that such a t exists, $t \geq 3$, and that $j_t > j_1$.) Let the F-simple commutator d(c) be defined by

$$d(c) = (c_{j_t}, d_2, d_3, \ldots, d_n) \qquad (3.1)$$

where $d_2, d_3, \ldots, d_n$ is that rearrangement of $c_{j_1}, \ldots, c_{j_{t-1}}, c_{j_{t+1}}, \ldots, c_{j_n}$ for which $d_2 \leq \ldots \leq d_n$. Then the following holds in J

$$c = H_1 \, d(c) \, H_2 \qquad (3.2)$$

where the $H_j (j = 1,2)$ are words in J-simple commutators u_i such that (i) $1 < W(u_i) < n$; (ii) if $u_i = (v_1, \ldots, v_k)$, then $v_1, \ldots, v_k$ is a rearrangement of a subsequence of $c_{j_1}, \ldots, c_{j_n}$; (iii) $W(H_1 \, d(c) \, H_2) \geq n$ in F.

Proof. We proceed by induction on the place of c in our lexicographic ordering of basic commutators. Clearly, $W(c) = n \geq 3$. For $n = 3$, we have $c = (c_{j_1}, c_{j_2}, c_{j_3})$ with $(c_{j_1}, c_{j_3}) = 1$ in J and $c_{j_3} > c_{j_1}$. Thus equation (3.20)-I yields that in J

$$c = (c_{j_1}, c_{j_2})^{-1}(c_{j_3}, c_{j_2})(c_{j_3}, c_{j_2}, c_{j_1})$$

$$(c_{j_1}, c_{j_2})(c_{j_3}, c_{j_2})^{-1}. \qquad (3.3)$$

This shows that we have (3.2) with properties (i) and (ii) for $n = 3$. In order to show that the right hand side of (3.3) also satisfies property (iii), we let $a = (c_{j_1}, c_{j_2})$, $b = (c_{j_3}, c_{j_2})$, and $A = (c_{j_3}, c_{j_2}, c_{j_1})$. Then (3.3) becomes

$$c = a^{-1} b \, A \, a \, b^{-1}$$

and we need to show that $W(a^{-1} b \, A \, a \, b^{-1}) \geq 3$ in F. But in F

$$\begin{aligned} a^{-1} \, b \, A \, a \, b^{-1} &= (a^{-1} \, b \, A \, b^{-1} a) \cdot (a, b^{-1}) \\ &= R \cdot (a, b^{-1}) \end{aligned}$$

where $R = a^{-1} \, b \, A \, b^{-1} \, a$. Evidently by (2.3a)-I, $W(a, b^{-1}) \geq 4$ in F so that

$$W(a^{-1} b A a b^{-1}) = W(R \cdot (a, b^{-1})) \geq W(R) \geq 3$$

in F. This proves the result for $n = W(c) = 3$.

Next, suppose that we have already proven the result for all $c < c_k$ with $W(c_k) > 3$. We then proceed to the smallest commutator c which satisfies the hypothesis of our theorem and is such that $c \geq c_k$. (Thus $n = W(c) \geq W(c_k) > 3$.)

We now define $\delta = n - t$ and we treat two cases: $\delta > 0$ and $\delta = 0$. For $\delta > 0$, let us write $e = c^L$ and we note that, in J, $e = (c_{j_1}, \ldots, c_{j_{n-1}})$ is by the induction hypothesis a word of the form (3.2), i.e.,

$$e = H_{11}\, d(e)\, H_{12},$$

where H_{1j} $(j = 1,2)$ are words in J-simple basic commutators z_i satisfying (i) and (ii) of our conclusion, and $W(H_{11} d(e) H_{12}) \geq n - 1$ in F. In particular every z_i has $1 < W(z_i) < n - 1$ and also has $z_i^R \leq c_{j_{n-1}}$ (recall by hypothesis that $c_{j_1} < c_{j_t} \leq c_{j_{n-1}}$). Applying the trivial identity (3.19)-I to the computation of $c = (e, c_{j_n}) = (H_{11} d(e) H_{12}, c_{j_n})$, we find that in J

$$c = \Pi_1 \, (d(e), c_{j_n}) \, \Pi_2 \qquad (3.4)$$

where the Π_j $(j = 1,2)$ are words in z_i, (z_i, c_{j_n}), and $d(e)$. (Note $d(e)$ only occurs in Π_1.) Now $d(c) = (d(e), c_{j_n})$ by (3.1) since $\delta > 0$. Here we denote the word on the right hand side of (3.4) by

w and note that w is a word in the F-simple basic commutators $d(e)$, z_i, (z_i, c_{j_n}), and $d(c)$, i.e.,

$$w = \Pi_1 \, d(c) \, \Pi_2 \tag{3.4a}$$

where the $\Pi_j (j = 1,2)$ are as in (3.4). Moreover, we note that $W(z_i, c_{j_n}) \le n - 1$. So that either the F-simple commutators (z_i, c_{j_n}) are all J-simple or we may use the induction hypothesis to write those (z_i, c_{j_n}) which are not J-simple as words of the form (3.2) in J-simple commutators. After making the replacements for those (z_i, c_{j_n}) in (3.4a) which are not J-simple, we find that in J

$$c = H_1 \, d(c) \, H_2 \tag{3.5}$$

where we have demonstrated that the H_j $(j = 1,2)$ are words in J-simple commutators satisfying conditions (i) and (ii) of our conlusion.

To see that our expression for c in (3.5) also satisfies condition (iii) of our conclusion, we first note that the word w as given in (3.4a) has $W(w) \ge n$ in F. (This is true since $w = (H_{11} d(e) H_{12}, c_{j_n})$ in F and $W(H_{11} d(e) H_{12}) \ge n - 1$.) Thus if all the (z_i, c_{j_n}) satisfy Criterion 3, i.e., are J-simple, we are done. Otherwise let Σ be set of all the distinct F-simple basic commutators which occur in w as given by (3.4a). Let $U = (u_1, \ldots, u_m) \subseteq \Sigma$ be those (z_i, c_{j_n}) in (3.4a) which are not

J-simple and let $\{ v_1, \ldots, v_m \}$ be those words in J-simple commutators which are obtained by applying the induction hypothesis to the (z_i, c_{j_n}). Thus $u_i = v_i$ in J and $W(u_i) \leq W(v_i)$ in F for all $i = 1, \ldots, m$. Now to go from (3.4) to (3.5), we replace the u_i by the v_i in w as given by (3.4a), i.e., we apply the map $\Psi : u_i \longrightarrow v_i$. Thus since $w \in F_n$, Lemma 2.1 implies that

$$\Psi(w) = H_1 \, d(c) \, H_2$$

in (3.5) is such that $(H_1 \, d(c) \, H_2) \in F_n$, i.e., $W(H_1 d(c) \, H_2) \geq n$. This completes the proof of our result in the case $\delta > 0$.

It remains to consider the case $\delta = 0$ for which we write $(c^L)^L = A$, $(c^L)^R = c_{j_n} = g$, amd $c^R = h$. Then identity (3.20)-I gives

$$c = (A,g,h) = (A,g)^{-1}(A,h)^{-1}A^{-1}(h,g)$$

$$A(A,g)(A,h)(A,h,g)(h,g)^{-1}. \tag{3.6}$$

Either $(h,g) = 1$ in J or it is J-simple. Also all the commutators A, (A,g), and (A,h) are F-simple of weight $< n$. Hence any such commutator is either J-simple or can be written in J by the induction hypothesis as a word in J-simple commutators satisfying conditions (i), (ii), and (iii) of our conclusion. Letting

$$Z = (A, h, g)$$

in (3.6), we claim that is sufficient to prove that Z can be written in J as a word in J-simple commutators in the form

$$Z = H_{11}\ d(c)\ H_{12} \tag{3.7}$$

where the $H_{1j}(j = 1,2)$ are words in J-simple commutators, u_i, satisfying (i) and (ii) of our conclusion, and

$$W\ (H_{11}\ d(c)\ H_{12}) \geq n.$$

To see this, suppose we substitute (3.7) and the expressions in terms of J-simple commutators (for those among A, (A,g), and (A,h) which are not J-simple) into (3.6). This writes c in J as

$$c = H_1\ d(c)\ H_2 \tag{3.8}$$

where the $H_j(j = 1,2)$ are words in J-simple commutators. Clearly, conditions (i) and (ii) of our conclusion are satisfied by (3.8), according to the *induction hypothesis* and our claim.

For condition (iii), we note that (3.6) is an identity in F. Thus the right hand side of (3.6) has weight n in F since c is F-simple of weight n. Now let (h,g) = d, (A,g) = a, and (A,h) = b. Then (3.6) becomes

$$\begin{aligned} c &= a^{-1}\ b^{-1}\ A^{-1}\ d\ A\ a\ b\ Z\ d^{-1} \\ &= (a,b)\ (A\ a\ b,\ d^{-1})\ (d\ Z\ d^{-1}). \end{aligned} \tag{3.6a}$$

By (2.3a-I), $a \in F_{n-1}$, $b \in F_{n-1}$, $A \in F_{n-2}$, $Z \in F_n$, and $d \in F_2$. (Either d = 1 or W(d) = 2.) Let $\tilde{a}$, $\tilde{b}$, $\tilde{A}$, $\tilde{Z}$ be the words in J-simple commutators such that $a = \tilde{a}$, $b = \tilde{b}$, $Z = \tilde{Z}$ in J (i.e., $\tilde{Z} = H_{11} d(c)\ H_{12}$ as in (3.7)). We know by the induction hypothesis

and our claim for Z that $\tilde{a} \in F_{n-1}$, $\tilde{b} \in F_{n-1}$, $\tilde{A} \in F_{n-2}$, $\tilde{Z} \in F_n$.

Also $\tilde{d} \in F_2$ since d either is 1 in J or it is a J-simple commutator of weight 2. (Here $\tilde{d}$ = d if ≠ 1 in J but $\tilde{d}$ = 1 if d = 1 in J.) Furthermore, the mapping of $F \longrightarrow J$ is a homomorphism; also the J-simple commutators are free generators of the subgroup of F_2 which they generate by Corollary 4.1-I. Thus

$$\tilde{c} = (\tilde{a}, \tilde{b})\ (\tilde{A}\ \tilde{a}\ \tilde{b}, \tilde{d}^{-1})\ (\tilde{d}\ \tilde{Z}\ \tilde{d}^{-1})$$

is according to (2.3a)-I such that $(\tilde{a}, \tilde{b}) \in F_{2n-2} \subseteq F_n$(since $2n - 2 > n$ in our case $n > 3$), $(\tilde{A}\ \tilde{a}\ \tilde{b}, \tilde{d}^{-1}) \in F_n$, and $\tilde{d}\ \tilde{Z}\ \tilde{d}^{-1} \in F_n$. Thus W $(\tilde{c}) \geq n$ in F where $\tilde{c}$ is the unique word in J-simple commutators which = c in J. (Here $\tilde{c} = H_1\ d(c)\ H_2$ as in (3.8).)

Thus we now focus our attention on Z = (A,h,g), and prove the claim for Z as given by (3.7). As already mentioned, b = (A, h) is, by the induction hypothesis, a word, $\tilde{b}$, in J of the form (3.2) in d(b) and J-simple commutators, s_i, of weight < n-1 with all the special properties of our conclusion. Therefore Z = (b, g) can be written by the trivial identity (3.19)-I in J in the form

$$Z = (\tilde{b}, g) = H_{01}(\, d(b), g)\ H_{02} \tag{3.9}$$

where the H_{0j} (j = 1,2) are words in d(b), s_i, and (s_i, g). By the

induction hypothesis $W(\tilde{b}) \geq n-1$, hence (2.3a)-I implies that the right side of (3.9), $H_{01}(d(b),g)H_{02}$, has weight at least n in F.

Let us now examine (d(b), g). From (3.1) and the fact that $\delta = 0$,

$$d(b) = (c_{j_n}, e_2, e_3, \ldots, e_{n-1})$$

where $e_2 \leq e_3 \leq \ldots \leq e_{n-1}$ is a rearrangement of $c_{j_1}, c_{j_2}, \ldots, c_{j_{n-2}}$. (Evidently $c_{j_2} = e_2$, since $c_{j_1} > c_{j_2} \leq c_{j_3} \leq \ldots \leq c_{j_n}$.) We note from Lemma 3.5-I that $M(d(b),g) = d(c)$. Next, we apply Lemma 2.2 to write (d(b),g) as $\Pi_1\, d(c)\, \Pi_2$ in F. (Note that $g = c_{j_{n-1}} \geq e_2$ so that we are in Case I of Lemma 2.2 and also that if $g \geq e_{n-1}$, then $(d(b), g) = d(c)$ and we just then take $\Pi_1 = \Pi_2 = 1$.)

Let us also apply Lemma 2.2 to rewite those (s_i, g) in (3.9) which are not basic (i.e., $g < s_i^R$ so that Lemma 2.2 applies) in terms of F-simple commutators. Thus

$$(s_i, g) = \Pi_{i1}\, M(s_i, g)\, \Pi_{i2}$$

in F. (Here it does not matter which case of (2.2) we are in because $W(s_i, g) < n$.) Thus Π_1, Π_2, and $\Pi_{ij}(j = 1,2)$ are words in F-simple commutators, z_i, all of which have $1 < W(z_i) < n$ and satisfy condition (ii) of our conclusion. Substituting these into (3.9), we have that Z can be written in J as a word, w , in F-simple commutators d(c), s_i, (s_i,g) (only those which are basic

are included here), and z_i, i.e., in J we have

$$Z = w = \Omega_{01}\ d(c)\ \Omega_{02} \tag{3.10}$$

where Ω_{0j} ($j = 1,2$) are words in the s_i, (s_i,g) -- only the basic ones -- , and z_i. We may now use the induction hypothesis to write those F-simple commutators, the (s_i, g) and z_i, which occur in (3.10) and are not J-simple, as words in J-simple commutators. Making these replacements in (3.10) gives (3.7). To see this, we note that now Z is written, in J, in terms of J-simple commutators, u_i, which clearly satisfy (i) and (ii) of our conclusion.

To see that $W(H_{11}d(c)\ H_{12}) \geq n$ in F, we first note that the right hand side of (3.9) is equal to the right hand side of (3.10) in F, i.e.,

$$w = H_{01}\ (d(b),\ g)\ H_{02}$$

holds in F. Thus $w \in F_n$. We can now continue in a manner exactly analogous to that used in the case of $\delta > 0$, because the situation in (3.10) is similar to that of equation (3.4a). In this way Lemma 2.1 shows that the weight of $H_{11}d(c)\ H_{12}$ given in (3.7) is at least n. This completes the proof of our claim for Z given in (3.7) and with that the proof of our theorem is accomplished.

CONCLUSION

Theorem 3.1 of this paper shows that in a group which is a free product of torsion free Abelian groups, here denoted by J, any F-simple commutator can be expressed in terms of J-simple commutators. It then follows from well known results [1] that the J-basic commutators of weight n form a basis for the quotient group J_n / J_{n+1}. Thus Theorem 3.1 completely determines these quotients.

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